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AN IDENTIFICATION PROBLEM ARISING IN THE THEORY OF HEAT CONDUCTION FOR MATERIALS WITH MEMORY

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Abstract - We deal with the problem of recovering a memory kernel $k(t,\eta)$, depending on time t and on a spatial variable η, in a parabolic integro-differential equation related to a bounded domain $\Omega \subset \mathbf{R}^3$. We show that, under suitable assumptions and two pieces of additional information, our identification problem can be uniquely solved locally in time.

1. INTRODUCTION

We deal with an identification problem arising when the classical theory of heat conduction is modified in order to describe the thermal behaviour of viscoelastic materials such as polymers, polymer solutions and suspensions (cf. [9]). With respect to metals, the mechanical properties of viscoelastic materials can be affected by the previous history, that is, by the method of fabrication, post-treatment, and the age of the finished article. This is why these materials are said to possess *memory*. Such a memory evinces in the constitutive relationship between the stress and the strain tensor and leads to mathematical models of viscoelastic phenomena which take the form of partial differential Volterra equations (cf. [1], [6] and [10]).

Indeed, if $\Omega \subset \mathbf{R}^3$ is a nonhomogeneous thermal body made of material with memory, then the variation of the temperature u with respect to time satisfies the following parabolic integro-differential equation, where the symbol "$*$" stands for the convolution operator $(v * w)(t) = \int_0^t v(t-s)w(s)\,\mathrm{d}s$:

$$D_t u(t,x) = \mathcal{A}u(t,x) + \mathrm{div}\left[\left(k(\cdot,\rho(x)) * \widetilde{b}(x)\nabla u(\cdot,x)\right)(t)\right]\mathrm{d}s + f(t,x)\,, \quad t > 0,\ x \in \Omega. \tag{1}$$

Here $\mathcal{A}$ is a second-order linear differential operator, k represents the *memory kernel* keeping the history record of the material, $\rho : \Omega \to \mathbf{R}$ is an assigned function, f represents the system of external heat sources, $\widetilde{b}(x)$ denotes a 3×3 matrix $(\widetilde{b}_{i,j}(x))_{i,j=1}^3$ and, finally, $\mathrm{div} = \sum_{i=1}^3 D_{x_i}$ and $\nabla = (D_{x_1}, D_{x_2}, D_{x_3})$.

In this paper, we will deal with the problem of recovering k in equations of the type of (1). It should be stressed that the recovery of a memory kernel k depending on both time and space is a quite new problem, as far as first-order in time integro-differential equations are concerned. See, for instance, [2], [5] and [7] which, however, have to be considered one-dimensional in character. In [2] the kernel k depends on time and on only one space variable between the n variables of $\mathbf{R}^n$, $n \geq 2$, whereas in [5] and [7] the kernel is assumed to be *degenerate*, i.e. of the form $k(t,x) = \sum_{j=1}^N m_j(t)\mu_j(x)$, but with the space-dependent functions μ_j, $j = 1,\ldots,N$, assumed to be *known*, too. As a consequence, the identification problem reduces to recovering the N unknown time-dependent functions m_j, $j = 1,\ldots,N$. Our aims will be those of generalizing these one-dimensional results, by skipping the assumption of degenerateness and searching for kernels which depend on both time and space, the spatial dependence occurring through scalar functions of *all* the variables at disposal.

We point out that the investigation of time and space dependent memory kernels seems to be very promising from the point of view of applications. For example, it gives a quite good description of the thermal behaviour of nonhomogeneous bodies and highlights the existence of unexpected symmetry relationships, between the analytic properties of kernels and the geometric structure of materials (cf. [3]).

2. FORMULATION OF THE IDENTIFICATION PROBLEM

Our problem is concerned with the identification of the unknown memory kernel k, depending on two scalar variable t, η, appearing in the following integro-differential equation of parabolic type related to a bounded domain $\Omega \subset \mathbf{R}^3$, where $(t,x) \in [0,T] \times \Omega$:

$$D_t u(t,x) = \mathcal{A}u(t,x) + [k(\cdot,\rho(x)) * \mathcal{B}u(\cdot,x)](t) + [D_\eta k(\cdot,\rho(x)) * \mathcal{C}u(\cdot,x)](t) + f(t,x)\,. \tag{2}$$

Here, $\mathcal{A}$ and $\mathcal{B}$ are two second-order linear differential operators, while $\mathcal{C}$ is a first-order differential

operator, having respectively the following forms:

$$\mathcal{A} = \sum_{j,k=1}^{3} D_{x_j}\left(a_{j,k}(x)D_{x_k}\right), \quad \mathcal{B} = \sum_{j,k=1}^{3} D_{x_j}\left(b_{j,k}(x)D_{x_k}\right), \quad \mathcal{C} = \sum_{j=1}^{3} c_j(x)D_{x_j}. \tag{3}$$

In addition, the operator $\mathcal{A}$ is uniformly elliptic whereas the function ρ satisfies the following assumptions:

(A) $\rho : V \to \rho(V) \subset \mathbf{R}$, where V is an open (possibly unbounded) set of $\mathbf{R}^3$ and $\rho(V) = (l_1, l_2)$, $-\infty \leq l_1 < l_2 \leq +\infty$;

(B) if $\overline{V} = V \cup \partial V$ denotes the $\mathbf{R}^3$–closure of V, then $\partial V = \bigcup_{i=1}^{2} \overline{U}_i$ where $U_i = \{x \in \mathbf{R}^3 : (-1)^i \limsup_{y \to x, \, y \in V} (-1)^i \rho(y) = l_i\}$, $i = 1, 2$;

(C) $\rho \in C^1(V)$ and $\nabla\rho(x) \neq 0$ for every $x \in V$;

(D) for any $y \in \partial V$ there exist $r_y > 0$ and at least an index $j(y) \in \{1, 2, 3\}$ such that $D_{x_{j(y)}}\rho(z) \neq 0$ for every $z \in V \cap (B(y, r_y) \backslash U_y^{j(y)})$, where $U_y^{j(y)}$ is a subset of $\mathbf{R}^3$ having three-dimensional Lebesgue measure equal to zero.

Observe that, in (B), one or both of the $\overline{U}_i$'s could be empty (for instance if $V = \mathbf{R}^3$) or they could coincide (for instance if ∂V is an hyperplane and V stands on both the sides of ∂V). Moreover, in (D), when $U_y^{j(y)}$ is contained in ∂V we can choose $U_y^{j(y)} = \emptyset$.

However, it is worth saying that assumptions (A)–(C), except for $\nabla\rho(\cdot) \neq 0$ in V and at least when $\overline{U}_1 \cap \overline{U}_2 = \emptyset$, are close to requiring $\rho \in C(\overline{V}) \cap C^1(V)$ and hence are quite natural. On the contrary, assumption (D) seems a very special one and hard to be satisfied. This is not completely true and, as a concrete example, we refer to [3, Examples 3.1.2] where it is shown that (A)–(D) are all satisfied by each one of the spherical coordinates (r, φ, θ), related to the Cartesian ones via the relationship $(x_1, x_2, x_3) = r(\cos\varphi\sin\theta, \sin\varphi\sin\theta, \cos\theta)$.

Coming back to our problem, we restrict our attention to any bounded domain Ω of $\mathbf{R}^3$ such that $\overline{\Omega} \subset \overline{V}$ and such that the boundary $\partial\Omega$ is the finite union of m pairwise disjoint surfaces $\partial\Omega_k$, $k = 1, \ldots, m$, of class C^2, i.e. $\partial\Omega = \bigcup_{k=1}^{m} \partial\Omega_k$, $\partial\Omega_i \cap \partial\Omega_j = \emptyset$ for any $i, j = 1, \ldots, m$, $i \neq j$. However, a bounded domain Ω of $\mathbf{R}^3$ satisfying the previous assumptions will be said an *admissible domain* for our problem if and only if the function ρ satisfies the following additional property on it:

(E) there exists a constant $\widetilde{C}$ such that $|\nabla\rho(x)| \leq \widetilde{C}$ for every $x \in \overline{\Omega} \cap V$.

Note that (E) is obvious if $\overline{\Omega} \subset V$, much less if $\partial\Omega \cap \partial V \neq \emptyset$. Now, $u_0 : \overline{\Omega} \to \mathbf{R}$ and $u_1 : [0, T] \times \overline{\Omega} \to \mathbf{R}$ being two assigned smooth functions, we prescribe the *initial condition*

$$u(0, x) = u_0(x), \quad \forall\, x \in \Omega, \tag{4}$$

as well as one of the following boundary conditions, where, for any $s \in \{1, \ldots, m\}$, the indexes i_k, $k = 1, \ldots, s$, satisfy $0 \leq i_1 < \ldots < i_s \leq m$ with the convention that $\partial\Omega_{i_1} = \emptyset$ when $i_1 = 0$:

$$(\mathrm{D}; i_1, \ldots, i_s) \qquad u(t, x) = u_1(t, x), \quad \forall\, (t, x) \in [0, T] \times \bigcup_{k=1}^{s} \partial\Omega_{i_k}, \tag{5}$$

$$(\mathrm{N}; i_1, \ldots, i_s) \qquad D_\nu u(t, x) = D_\nu u_1(t, x), \quad \forall\, (t, x) \in [0, T] \times (\partial\Omega \backslash \bigcup_{k=1}^{s} \partial\Omega_{i_k}). \tag{6}$$

Here ν stands for the conormal vector $\nu = (\nu_1, \nu_2, \nu_3)$ being defined by $\nu_j(x) = \sum_{k=1}^{3} a_{j,k}(x)n_k(x)$, where $n(x) = (n_1(x), n_2(x), n_3(x))$ denotes the unit outer normal vector at $x \in \partial\Omega$.

Having to deal with the inverse problem of determining both u and k in (2), it seems to be reasonable to prescribe one more additional information, to be used for reconstructing k, and to hope that only one may be enough. By the way, this is not the case, since the kernel k we want to recover explicitly depends on the variables t and $\eta = \rho(x)$, $x \in \Omega \cap V$, and therefore the problem is two-dimensional in character. Indeed, by setting

$$\begin{cases} l_3 = \inf_{x \in \Omega \cap \overline{V}}\{\liminf_{y \to x, \, y \in \Omega \cap V} \rho(y)\}, \\ l_4 = \sup_{x \in \Omega \cap \overline{V}}\{\limsup_{y \to x, \, y \in \Omega \cap V} \rho(y)\}, \end{cases} \tag{7}$$

fixing $\eta_0 \in [l_3, l_4]$ and assuming $k(t, \eta)$ differentiable with respect to η, from the fundamental formula of Integral Calculus we get

$$k(t, \eta) = h(t) + \int_{\eta_0}^{\eta} q(t, \xi) \, \mathrm{d}\xi =: h(t) + Eq(t, \eta), \quad \forall (t, \eta) \in [0, T] \times (l_3, l_4), \tag{8}$$

where h and q denote the new unknown functions

$$h(t) = k(t, \eta_0), \qquad q(t, \eta) = D_\eta k(t, \eta). \tag{9}$$

Consequently, recovering k is equivalent to recovering h and q and, as we will see further, the original problem of identifying the pair (u, k) in equation (2) can be reformulated as an equivalent one related to the triplet (v, h, q) where $v = D_t u - D_t u_1$. Hence, it clearly appears that one additional piece of information does not suffice to solve our identification problem, and, by virtue of decomposition (8), at least two additional measurements have to be available. Therefore, we assume also to be given the two following additional pieces of information

$$\Phi[u(t, \cdot)](\eta) := g_1(t, \eta), \qquad \forall (t, \eta) \in [0, T] \times (l_3, l_4), \tag{10}$$

$$\Psi[u(t, \cdot)] := g_2(t), \qquad \forall t \in [0, T], \tag{11}$$

where Φ, Ψ and g_i, $i = 1, 2$, are, respectively, given linear operators acting on spatial variables and smooth prescribed functions.

By writing $\omega = \omega(\cdot)$ when this shortening does not generate any confusion, from (4)–(6), (10), (11) we (formally) deduce that our data have to satisfy also the following consistency conditions

$$(\mathrm{C1}, \mathrm{D}, \mathrm{N}) \qquad u_0 = u_1(0, \cdot) \quad \text{on } \Gamma_\mathrm{D} \quad \text{and} \quad D_\nu u_0 = D_\nu u_1(0, \cdot) \quad \text{on } \Gamma_\mathrm{N} \tag{12}$$

$$\Phi[u_0](\eta) = g_1(0, \eta), \qquad \Psi[u_0] = g_2(0), \tag{13}$$

Γ_D and Γ_N in (12) denoting, respectively, those parts of the boundary of Ω where Dirichlet and Neumann conditions are possibly prescribed.

Convention: From now on, when Ω is an admissible domain, we will denote by $\mathrm{P}(\mathrm{D}, \mathrm{N})$ the identification problem consisting of (2), (4), the boundary conditions (5), (6) and the additional pieces of information (10), (11).

3. MAIN ASSUMPTIONS AND BASIC SYSTEM

Since we are interested in an existence and uniqueness result to the identification problem $\mathrm{P}(\mathrm{D}, \mathrm{N})$, the working technique seems to be that of reformulating our problem in a Banach space framework and then to apply a fixed-point argument to such a problem (cf. [2]).

To perform this procedure, we first rewrite our problem for the pair (u, k) in an equivalent one for the triplet (v, h, q), where h and q are defined by (9) whereas v is defined via the following formula

$$v := D_t u - D_t u_1 \quad \Longleftrightarrow \quad u(t, x) = u_1(t, x) - u_1(0, x) + u_0(x) + \int_0^t v(s, x) \, \mathrm{d}s, \tag{14}$$

u_0 and u_1 being, respectively, the prescribed initial and boundary data (4)–(6).

To obtain the basic system for the triplet (v, h, q) we need some further assumptions, additional to the linearity, for the operators Φ and Ψ appearing in (10) and (11). We now list the essential ones. Let Ξ be the set of spatial variables on which Φ acts and let $v_1 : (l_3, l_4) \to \mathbf{R}$, $v_2 : \Omega \to \mathbf{R}$ and $v_3 : (l_3, l_4) \times \Xi \to \mathbf{R}$ be three functions for which at the moment we do not require any other regularity but that v_3 admits (generalized) derivative with respect to $\eta \in (l_3, l_4)$. Then the following relations hold:

$$\Phi[v_1 v_2] = v_1 \Phi[v_2], \qquad D_\eta \Phi[v_3](\eta) = \Phi[D_\eta v_3](\eta), \quad \forall \eta \in (l_3, l_4). \tag{15}$$

It occurs that (15) is not sufficient and we need also the two *basic* decomposition formulae

$$\Phi \mathcal{A} = \mathcal{A}_1 \Phi + \Phi_1, \qquad \Psi \mathcal{A} = \Psi_1, \tag{16}$$

where Φ_1 and Ψ_1 are two linear operators and $\mathcal{A}_1$ is a differential operator $\mathcal{A}_1(\eta; D_\eta)$.

Further, we assume that u_0 satisfies the following conditions for some positive constant m_0:

$$J_0(u_0)(\eta) \; := \; \big|\Phi[\mathcal{C}u_0](\eta)\big| \geq m_0\,, \qquad \forall\, \eta \in (l_3, l_4), \tag{17}$$

$$J_1(u_0) \; := \; \Psi[J(u_0)] \neq 0\,, \tag{18}$$

where we have set:

$$J(u_0)(x) := \Big(\mathcal{B}u_0(x) - \frac{\Phi[\mathcal{B}u_0](\rho(x))}{\Phi[\mathcal{C}u_0](\rho(x))}\mathcal{C}u_0(x)\Big) \exp\Big[\int_{\rho(x)}^{\eta_0} \frac{\Phi[\mathcal{B}u_0](\xi)}{\Phi[\mathcal{C}u_0](\xi)}\,\mathrm{d}\xi\Big], \quad \forall\, x \in \Omega\,.$$

Now, let us suppose that the pair (u, k) is a solution to the identification problem $P(D, N)$ such that u is twice differentiable with respect to time and k is once differentiable with respect to η. Replacing k in (2) with the right-hand side of (8) and differentiating the so obtained equation with respect to time, we deduce that the triplet (v, h, q) satisfies the following identity for any $(t, x) \in [0, T] \times \Omega$:

$$
\begin{aligned}
D_t v(t, x) = \mathcal{A}v(t, x) &+ \Big(\big[h(\cdot) + Eq(\cdot, \rho(x))\big] * \big[\mathcal{B}v(\cdot, x) + \mathcal{B}D_t u_1(\cdot, x)\big]\Big)(t) \\
&+ \big[h(t) + Eq(t, \rho(x))\big]\mathcal{B}u_0(x) + \Big(q(\cdot, \rho(x)) * \big[\mathcal{C}v(\cdot, x) + \mathcal{C}D_t u_1(\cdot, x)\big]\Big)(t) \\
&+ q(t, \rho(x))\mathcal{C}u_0(x) + (\mathcal{A} - D_t)D_t u_1(t, x) + D_t f(t, x)\,.
\end{aligned}
\tag{19}
$$

Moreover, evaluating (2) in $t = 0$ and taking advantage of (4) we deduce that v satisfies the initial condition

$$v(0, x) = \mathcal{A}u_0(x) + f(0, x) - D_t u_1(0, x) =: v_0(x)\,, \qquad \forall\, x \in \Omega\,, \tag{20}$$

whereas, concerning the boundary values, from (14) and (5), (6) we get

$$v(t, \cdot) = D_\nu v(t, \cdot) = 0\,, \quad \text{on } \partial\Omega\,, \quad \forall\, t \in [0, T].\tag{21}$$

Finally, since Φ and Ψ act on spatial variables, only, they commute with the derivative with respect to time D_t. Consequently, from (10) and (11) we get

$$\Phi[v(t, \cdot)](\eta) = D_t g_1(t, \eta) - \Phi[D_t u_1(t, \cdot)](\eta)\,, \qquad \forall\,(t, \eta) \in [0, T] \times (l_3, l_4)\,, \tag{22}$$

$$\Psi[v(t, \cdot)] = D_t g_2(t) - \Psi[D_t u_1(t, \cdot)]\,, \qquad \forall\, t \in [0, T].\tag{23}$$

Now, the consistency conditions related to problem (19)–(23) are explicitly given by:

$$(\mathrm{C2, D, N}) \qquad v_0 = 0 \quad \text{on } \Gamma_{\mathrm{D}} \qquad \text{and} \qquad D_\nu v_0 = 0 \quad \text{on } \Gamma_{\mathrm{N}} \tag{24}$$

$$\Phi[v_0](\eta) = D_t g_1(0, \eta) - \Phi[D_t u_1(0, \cdot)](\eta)\,, \qquad \Psi[v_0] = D_t g_2(0) - \Psi[D_t u_1(0, \cdot)]\,, \tag{25}$$

Having derived the fundamental equations for v, we now turn our attention to h and q. Applying Φ and Ψ to both sides of (19) and taking advantage of (15) and (16), it is easy to check that, for any $(t, \eta) \in [0, T] \times (l_3, l_4)$, the following equations hold true:

$$
\begin{aligned}
q(t, \eta)\Phi[\mathcal{C}u_0](\eta) + Eq(t, \eta)\Phi[\mathcal{B}u_0](\eta) \;=\; &N_1^0(u_1, g_1, f)(t, \eta) + \Phi[N_1(v, h, q)(t, \cdot)](\eta) \\
&- \Phi_1[v(t, \cdot)](\eta) - h(t)\Phi[\mathcal{B}u_0](\eta)\,,
\end{aligned}
\tag{26}
$$

$$
\begin{aligned}
\Psi[q(t, \cdot)\mathcal{C}u_0 + Eq(t, \cdot)\mathcal{B}u_0] \;=\; &N_2^0(u_1, g_2, f)(t) + \Psi[N_1(v, h, q)(t, \cdot)] \\
&- \Psi_1[v(t, \cdot)] - h(t)\Psi[\mathcal{B}u_0].
\end{aligned}
\tag{27}
$$

where, for any $(t, \eta, x) \in [0, T] \times (l_3, l_4) \times \Omega$, operators N_1^0, N_2^0 and N_1 are defined, respectively, by formulae (3.2.19), (3.2.20) in [3] and

$$
\begin{aligned}
N_1(v, h, q)(t, x) = &-\Big(\big[h(\cdot) + Eq(\cdot, \rho(x))\big] * \big[\mathcal{B}v(\cdot, x) + \mathcal{B}D_t u_1(\cdot, x)\big]\Big)(t) \\
&+ \Big(q(\cdot, \rho(x)) * \big[\mathcal{C}v(\cdot, x) + \mathcal{C}D_t u_1(\cdot, x)\big]\Big)(t)\,.
\end{aligned}
\tag{28}
$$

Remark 3.1. Now, suppose that the triplet (v, h, q) solves the identification problem (19)–(23), (26) and (27). Then, performing integrations with respect to time and taking advantage of the consistency conditions (12), (13), it is easy to show that the pair (u, k), related to (v, h, q) by (9) and (14), solves our original problem $P(D, N)$. Hence, problem $P(D, N)$ and problem (19)–(23), (26), (27) are *equivalent*.

Now, due to (8), to determine $k(t, \cdot)$ for any $t \in [0, T]$ we have to solve system (26), (27) for h and q. This part involves a lot of definition that will probably takes out of our purposes. Therefore, for brevity, here we limit ourselves to refer the reader to [3, Section 3.2] and to recall only that the pair (h, q) solves the following fixed-point equations:

$$h(t) = h_0(t) + N_3(v, h, q)(t), \tag{29}$$

$$q(t, \eta) = q_0(t, \eta) + J_2(u_0)(\eta) N_3(v, h, q)(t) + N_2(v, h, q)(t, \eta). \tag{30}$$

Here h_0 and q_0 are defined, respectively, by formulae (3.2.38) and (3.2.42) in [3] whereas

$$N_2(v, h, q)(t, \eta) = J_3(u_0)\big\{\Phi[N_1(v, h, q)(t, \cdot)](\eta) - \Phi_1[v(t, \cdot)](\eta)\big\},$$

$$N_3(v, h, q)(t) := [J_1(u_0)]^{-1}\Big\{\Psi[N_1(v, h, q)(t, \cdot)] - \Psi[N_2(v, h, q)(t, \cdot)\mathcal{C}u_0]$$
$$+ \Psi[E\big(N_2(v, h, q)(t, \cdot)\big)\mathcal{B}u_0] - \Psi_1[v(t, \cdot)]\Big\},$$

$$J_2(u_0)(\eta) := -\frac{\Phi[\mathcal{B}u_0](\eta)}{\Phi[\mathcal{C}u_0](\eta)} \exp\Big[\int_\eta^{\eta_0} \frac{\Phi[\mathcal{B}u_0](\xi)}{\Phi[\mathcal{C}u_0](\xi)}\, d\xi\Big],$$

the linear operator $J_3(u_0)$ appearing in N_2 being that defined by formula (3.2.30) in [3]. We can now summarize the result of this section in the following equivalence theorem.

Theorem 3.2. *The pair (u, k) is a solution to the identification problem $P(D, N)$ if and only if the triplet (v, h, q) defined by (9) and (14) solves the problem (19)–(23), (29), (30).*

4. AN ABSTRACT REFORMULATION OF THE PROBLEM (19)–(23), (29), (30)

Starting from the result of Section 3, we now reformulate our identification problem in a Banach space framework. Let X be a complex Banach space with norm $\|\cdot\|_X$ and let $A : \mathcal{D}(A) \subset X \to X$ be a linear operator, with a non–necessarily dense domain, satisfying the following assumption:

(H1) *The resolvent $\rho(A)$ of A contains the half-plane $S_0 = \{\lambda \in \mathbf{C} : \mathrm{Re}\lambda \geq 0\}$ and there exists $M_0 > 0$ such that $\|(zI - A)^{-1}\|_{\mathcal{L}(X)} \leq M_0|1 + z|^{-1}$ for every $z \in S_0$.*

As it is well-known, assumption (H1) implies that A generates an analytic semigroup $\{e^{tA}\}_{t \geq 0}$ satisfying the estimates $\|A^k e^{tA}\|_{\mathcal{L}(X)} \leq \widetilde{c}_k t^{-k}$, $t > 0$, $k \in \mathbf{N} \cup \{0\}$. Moreover, from now on, we denote by $X_{1/2}$ the intermediate space $\mathcal{D}_A(1/2, p)$, $p \in [1, +\infty]$ between $\mathcal{D}(A)$ and X (cf. [8]).

In order to reformulate in an abstract form the problem (19)–(23), (29), (30) we need the following list of assumptions involving spaces, operators and data, where $0 < \beta < \alpha < 1/2$ and $\widetilde{q}_0$ is defined in Remark 3.3.3 in [3]:

(H2) Y, Y_1, $\mathcal{D}(B)$, $\mathcal{D}(C)$ *are Banach spaces such that $Y_1 \hookrightarrow Y$ and $\mathcal{D}(A) \hookrightarrow \mathcal{D}(B) \hookrightarrow \mathcal{D}(C) \hookrightarrow X$, $X_{1/2} \hookrightarrow \mathcal{D}(C)$;*

(H3) $B : \mathcal{D}(B) \to X$ *and* $C : \mathcal{D}(C) \to X$ *are linear operators such that $BA^{-1} \in \mathcal{L}(X)$ and $CA^{-1} \in \mathcal{L}(X; \mathcal{D}(C))$;*

(H4) $E \in \mathcal{L}(Y; Y_1)$, $\Phi \in \mathcal{L}(X; Y)$, $\Phi_1 \in \mathcal{L}(\mathcal{D}(C); Y)$, $\Psi \in X^*$, $\Psi_1 \in \mathcal{D}(C)^*$;

(H5) $\mathcal{M}$ *is a continuous bilinear operator from $Y \times \mathcal{D}(C)$ to X and from $Y_1 \times X$ to X;*

(H6) $J_1 : \mathcal{D}(B) \to \mathbf{R}$, $J_2 : \mathcal{D}(B) \to Y$, $J_3 : \mathcal{D}(B) \to \mathcal{L}(Y)$, *are three prescribed (nonlinear) operators;*

(H7) $u_0 \in \mathcal{D}(B)$, $Cu_0 \in \mathcal{D}(C)$, $J_1(u_0) \neq 0$, $Bu_0 \in \mathcal{D}_A(\alpha, +\infty)$;

(H8) $v_0 \in \mathcal{D}(A)$, $h_0 \in C^\beta([0,T]; \mathbf{R})$, $q_0 \in C^\beta([0,T]; Y)$;

(H9) $z_0 \in C^\beta([0,T]; X)$, $z_1 \in C^\beta([0,T]; \mathcal{D}(C))$, $z_2 \in C^\beta([0,T]; X)$;

(H10) $Av_0 + \mathcal{M}(\widetilde{q}_0, Cu_0) - \mathcal{M}(E\widetilde{q}_0, Bu_0) + z_2(0, \cdot) \in \mathcal{D}_A(\beta, +\infty)$;

Now (cf. (19)), denoting by $\mathcal{K}$ the convolution operator $\mathcal{K}(\chi, \kappa)(t) := \int_0^t \mathcal{M}\big(\chi(t-s), \kappa(s)\big) \, \mathrm{d}s$, our direct problem depending on the pair of parameter (h, q), is the following: *determine a function* $v \in C^1([0,T]; X) \cap C([0,T]; \mathcal{D}(A))$ *satisfying*

$$
\begin{cases}
v'(t) = [\lambda_0 I + A]v(t) + [h * (Bv + z_0)](t) - \mathcal{K}(Eq, Bv + z_0)(t) + \mathcal{M}(q(t), Cu_0) \\
\qquad + h(t)Bu_0 + \mathcal{K}(q, Cv + z_1)(t) - \mathcal{M}(Eq(t), Bu_0) + z_2(t), \quad \forall t \in [0,T], \\
v(0) = v_0.
\end{cases}
\tag{31}
$$

Remark 4.1. In the explicit case (19), we have $A = \mathcal{A} - \lambda_0 I$, with a large enough positive λ_0, and $z_0 = \mathcal{B}D_t u_1$, $z_1 = \mathcal{C}D_t u_1$, $z_2 = (\mathcal{A} - D_t)D_t u_1 + D_t f$. Instead, functions v_0, h_0 and q_0 in (H8) are those appearing, respectively, in (20), (29) and (30).

We now introduce the following unknown function w:

$$
w = Av \qquad \Longleftrightarrow \qquad v = A^{-1}w.
$$

Hence, applying A to the Volterra operator equation equivalent to problem (31), we obtain that w solves the equation

$$
w = w_0 + R_1(w, h, q) + S_1(q),
\tag{32}
$$

where we have set

$$
w_0 := Ae^{tA}v_0 + A(e^{tA} * z_2),
$$

$$
R_1(w, h, q) := \lambda_0(e^{tA} * w) + A[e^{tA} * h * (BA^{-1}w + z_0)] + A(h * e^{tA}Bu_0)
$$
$$
\qquad - A[e^{tA} * \mathcal{K}(Eq, BA^{-1}w + z_0)] + A[e^{tA} * \mathcal{K}(q, CA^{-1}w + z_1)],
$$

$$
S_1(q) := A[e^{tA} * (\mathcal{M}(q, Cu_0) - \mathcal{M}(Eq, Bu_0))].
$$

Now we rewrite the fixed-point system (29), (30) in an abstract form. For this purpose we introduce the operators:

$$
R_2(w, h, q) := -\widetilde{J_1}(u_0)\big\{ \Psi\big[\mathcal{M}(J_3(u_0)\Phi[N_1(A^{-1}w, h, q)], Cu_0) - N_1(A^{-1}w, h, q)
$$
$$
\qquad - \mathcal{M}(EJ_3(u_0)\Phi[N_1(A^{-1}w, h, q)], Bu_0)\big]\big\},
$$

$$
R_3(w, h, q) := J_2(u_0)R_2(w, h, q) + J_3(u_0)\Phi[N_1(A^{-1}w, h, q)],
$$

$$
S_2(w) := \widetilde{J_1}(u_0)\big\{ \Psi\big[\mathcal{M}(J_3(u_0)\Phi_1[A^{-1}w], Cu_0) - \mathcal{M}(EJ_3(u_0)\Phi_1[A^{-1}w], Bu_0)\big] - \Psi_1[A^{-1}w]\big\},
$$

$$
S_3(w) := J_2(u_0)S_2(w) - J_3(u_0)\Phi_1[A^{-1}w],
$$

where (cf. (H7) and (28)) we have set $\widetilde{J_1}(u_0) = [J_1(u_0)]^{-1}$ and

$$
N_1(A^{-1}w, h, q) = -h * (BA^{-1}w + z_0) + \mathcal{K}(Eq, BA^{-1}w + z_0) - \mathcal{K}(q, CA^{-1}w + z_1).
$$

Then, the fixed-point system for h and q can be rewritten in the following more compact form:

$$
h = h_0 + R_2(w, h, q) + S_2(w), \qquad q = q_0 + R_3(w, h, q) + S_3(w).
\tag{33}
$$

h_0 and q_0 being the elements appearing in (H8). Now, the fixed-point system (32) and (33) coincide with that in Section 5 of [2]. Therefore, referring the reader to Section 6 of [2] for the proof, we can state the following local in time existence and uniqueness result.

Theorem 4.2. *Under assumptions* (H1)–(H10) *there exists* $T^* \in (0, T)$ *such that for any* $\tau \in (0, T^*]$ *the fixed-point system* (32), (33) *has a unique solution* $(w, h, q) \in C^\beta([0, \tau]; X) \times C^\beta([0, \tau]; \mathbf{R}) \times C^\beta([0, \tau]; Y)$.

Corollary 4.3. *Under assumptions* (H1)–(H10) *there exists* $T^* \in (0, T)$ *such that for any* $\tau \in (0, T^*]$ *the problem* (31), (33) *admits a unique solution* $(v, h, q) \in \left[C^{1+\beta}([0, \tau]; X) \cap C^\beta([0, \tau]; \mathcal{D}(A))\right] \times C^\beta([0, \tau]; \mathbf{R}) \times C^\beta([0, \tau]; Y)$.

5. APPLICATION OF THE ABSTRACT RESULT TO P(D, N)

In this section, taking advantage of the equivalence Theorem 3.2 and the abstract Corollary 4.3, we solve (locally in time) the identification problem P(D, N), at least with the convention to work in the framework of Sobolev spaces related to $L^p(\Omega)$ with

$$p \in (3, +\infty). \tag{34}$$

We start by listing our main requirements on the operators Φ and Ψ. They are the same as in Section 3, but here we rewrite them in a more formal way, making clear the minimal space regularity required for the functions v_j, $j = 1, 2, 3$, and the operators Φ_1, Ψ_1 appearing on (15) and (16), respectively. Recalling that Ξ is the set of spatial variables on which Φ acts and taking into account (7), we assume:

$$\Phi \in \mathcal{L}(L^p(\Omega); L^p(l_3, l_4)), \qquad \Psi \in L^p(\Omega)^*, \tag{35}$$

$$\Phi[v_1 v_2] = v_1 \Phi[v_2], \quad \forall (v_1, v_2) \in L^p(l_3, l_4) \times L^p(\Omega), \tag{36}$$

$$D_\eta \Phi[v_3](\eta) = \Phi[D_\eta v_3](\eta), \quad \forall v_3 \in W^{1,p}((l_3, l_4) \times \Xi) \text{ and } \eta \in (l_3, l_4), \tag{37}$$

$$\Phi \mathcal{A} = \mathcal{A}_1 \Phi + \Phi_1 \text{ on } W_{\mathrm{D,N}}^{2,p}(\Omega), \qquad \Phi_1 \in \mathcal{L}(W^{1,p}(\Omega); L^p(l_3, l_4)), \tag{38}$$

$$\Psi \mathcal{A} = \Psi_1 \text{ on } W_{\mathrm{D,N}}^{2,p}(\Omega), \qquad \Psi_1 \in W^{1,p}(\Omega)^*, \tag{39}$$

where $\mathcal{A}_1$ is a second-order differential operator $\mathcal{A}_1 = \mathcal{A}_1(D_\eta)$ and $W_{\mathrm{D,N}}^{k,s}(\Omega)$, $k \in \mathbf{N}$, $s \in [1, +\infty]$, denote the space of function $\omega \in W^{k,s}(\Omega)$ satisfying the homogeneous condition (D,N).

To state our result concerning the identification problem P(D, N) we need to list also the following assumptions on the data f, u_j, g_k, $j = 0, 1$, $k = 1, 2$:

$$f \in C^{1+\beta}([0, T]; L^p(\Omega)), \quad f(0, \cdot) \in W^{2,p}(\Omega), \tag{40}$$

$$u_1 \in C^{2+\beta}([0, T]; L^p(\Omega)) \cap C^{1+\beta}([0, T]; W^{2,p}(\Omega)), \tag{41}$$

$$u_0 \in W^{4,p}(\Omega), \quad v_0 \in W_{\mathrm{D,N}}^{2,p}(\Omega), \quad \mathcal{B} u_0 \in W_{\mathrm{D,N}}^{2\alpha,p}(\Omega), \tag{42}$$

$$F := k_0' \mathcal{C} u_0 + k_0 \mathcal{B} u_0 + \mathcal{A} v_0 + (\mathcal{A} - D_t) D_t u_1(0, \cdot) + D_t f(0, \cdot) \in W_{\mathrm{D,N}}^{2\beta,p}(\Omega), \tag{43}$$

$$g_1 \in C^{2+\beta}([0, T]; L^p(l_3, l_4)) \cap C^{1+\beta}([0, T]; W^{2,p}(l_3, l_4)), \tag{44}$$

$$\mathcal{A}_1 D_t g_1 \in C^\beta([0, T]; L^p(l_3, l_4)), \tag{45}$$

$$g_2 \in C^{2+\beta}([0, T]; \mathbf{R}), \tag{46}$$

where $0 < \beta < \alpha < 1/2$, $\alpha, \beta \neq 1/(2p)$, and functions v_0 and k_0 appearing in (42) and (43) are defined, respectively, by formulae (20) and (3.18) in [4] (with the triplet (r, R_2, η) being replaced by (η, η_0, σ)). Here the spaces $W_{\mathrm{D,N}}^{2\gamma,p}(\Omega) \equiv (L^p(\Omega), W_{\mathrm{D,N}}^{2,p}(\Omega))_{\gamma,p}$, $\gamma \in (0, 1/2] \backslash \{1/(2p)\}$, are interpolation spaces between $W_{\mathrm{D,N}}^{2,p}(\Omega)$ and $L^p(\Omega)$ and they are defined (cf. [11, Section 4.3.3]) by:

$$W_{\mathrm{D,N}}^{2\gamma,p}(\Omega) = \begin{cases} W^{2\gamma,p}(\Omega), & \text{if } 0 < \gamma < 1/(2p), \\ \{u \in W^{2\gamma,p}(\Omega): u = 0 \text{ on } \Gamma_{\mathrm{D}}\}, & \text{if } 1/(2p) < \gamma \leq 1/2, \\ W^{2\gamma,p}(\Omega), & \text{if } 0 < \gamma \leq 1/2 \text{ and } \Gamma_{\mathrm{D}} = \emptyset. \end{cases} \tag{47}$$

Further, we introduce the Banach spaces $\mathcal{U}^{s,\beta,p}(T)$, $s \in \mathbf{N} \backslash \{0\}$, defined by

$$\mathcal{U}^{s,\beta,p}(T) = C^{s+\beta}([0, T]; L^p(\Omega)) \cap C^{s-1+\beta}([0, T]; W^{2,p}(\Omega)). \tag{48}$$

Analogously, replacing $W^{2,p}(\Omega)$ with $W^{2,p}_{\mathrm{D,N}}(\Omega)$ in (48), we define spaces $\mathcal{U}^{s,\beta,p}_{\mathrm{D,N}}(T)$.

Finally, we assume that the coefficients of operators $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ satisfy the following properties:

$$a_{i,j} \in W^{2,\infty}(\Omega), \quad a_{i,j} = a_{j,i}, \quad b_{i,j}, c_i \in W^{1,\infty}(\Omega), \qquad i,j = 1,2,3. \tag{49}$$

We can now state the main result of the section.

Theorem 5.1. *Let assumptions* (3), (49), (34)–(39) *be fulfilled and assume the function ρ satisfies the assumptions* (A)–(E) *of Section 2. Moreover, let the data vector (u_0, u_1, g_1, g_2, f) satisfies assumptions* (40)–(46), *inequalities* (17), (18) *and consistency conditions* (12), (13), *as well as* (24), (25).

Then, there exists $T^* \in (0,T]$ such that the identification problem $\mathrm{P(D,N)}$ *admits a unique solution $(u,k) \in \mathcal{U}^{2,\beta,p}(T^*) \times C^\beta([0,T^*]; W^{1,p}(l_3, l_4))$ depending continuously on the data with respect to the norms related to the Banach spaces in* (40)–(46).

To prove Theorem 5.1 we take advantage of the equivalence result of Section 3. Indeed, in the space framework above defined, Theorem 3.2 reads as follows.

Theorem 5.2. *The pair $(u,k) \in \mathcal{U}^{2,\beta,p}(T^*) \times C^\beta([0,T^*]; W^{1,p}(l_3, l_4))$ is a solution to the identification problem* $\mathrm{P(D,N)}$ *if and only if the triplet (v,h,q) defined by* (9) *and* (14) *belongs to $\mathcal{U}^{1,\beta,p}_{\mathrm{D,N}}(T^*) \times C^\beta([0,T^*]; \mathbf{R}) \times C^\beta([0,T^*]; L^p(l_3, l_4))$ and solves the problem* (19)–(23), (29), (30).

Consequently, it suffices to prove the following theorem, because then Theorem 5.1 trivially follows from Theorem 5.2.

Theorem 5.3. *Let the assumptions of Theorem 5.2 be fulfilled. Then, the identification problem* (19)–(23), (29), (30) *admits a unique solution $(v,h,q) \in \mathcal{U}^{1,p}_{\mathrm{D,N}}(T^*) \times C^\beta([0,T^*]; \mathbf{R}) \times C^\beta([0,T^*]; L^p(l_3, l_4))$ depending continuously on the data with respect to the norms related to the Banach spaces in* (40)–(46).

In order to prove Theorem 5.3 a preliminary result is needed. It deals with an estimate which, in the proof of Theorem 5.3, will allow us to prove assumption (H5) for the bilinear operator:

$$\mathcal{M} : \begin{cases} L^p(l_3, l_4) \times W^{1,p}(\Omega) \\ W^{1,p}(l_3, l_4) \times L^p(\Omega) \end{cases} \longrightarrow L^p(\Omega), \qquad \mathcal{M}(q,w)(x) = q(\rho(x))w(x), \quad x \in \Omega.$$

Lemma 5.4. *Let $\Omega \subset \mathbf{R}^3$ be a bounded domain satisfying the assumptions of Section 2 and let ρ be a function satisfying the assumptions* (A)–(E) *of the same section. Then, for every $s \in [1, +\infty)$, the following estimate holds*

$$\|g \circ \rho\|_{L^s(\Omega)} \le C_1(\rho, \Omega)\|g\|_{L^s(l_3, l_4)}, \qquad \forall g \in L^s(l_3, l_4), \tag{50}$$

where $C_1(\rho, \Omega)$ is a positive constant depending on ρ and Ω, only.

Proof. By recalling $\overline{\Omega} \subset \overline{V}$, we set $\Omega_1 = \overline{\Omega}\backslash\partial V$ and $\Omega_2 = \overline{\Omega} \cap \partial V$, so that $\Omega_1 \subset V$, $\Omega_2 \subset \partial V$ and $\overline{\Omega} = \Omega_1 \cup \Omega_2$. Now, since $|\nabla\rho(\cdot)|$ is positive in V, for every $y \in \Omega_1$ there exists $k = k(y) \in \{1,2,3\}$ such that $D_{x_k}\rho(y) \ne 0$. According to this, for every $y \in \Omega_1$, we denote by $i(y)$ the minimum index $k \in \{1,2,3\}$ such that $D_{y_k}\rho(y) \ne 0$ and by $B_y^{i(y)}$ the ball $B(y, r_y)$, $r_y > 0$, such that $D_{x_{i(y)}}\rho(z) \ne 0$ for every $z \in \overline{B(y, r_y)}$. This is possible since $\Omega_1 \subset V$ and V is open. Hence, at least by taking sufficiently small r_y, the ball $B_y^{i(y)}$ is well defined. Similarly (cf. (D)), for every $y \in \Omega_2$ there exist $r_y > 0$, $i(y) = \min\{1,2,3\}$ and a subset $U_y^{i(y)}$ of $\mathbf{R}^3$ having zero three-dimensional Lebesgue measure such that $D_{x_{i(y)}}\rho(z) \ne 0$ for every $z \in B(y, r_y)\backslash U_y^{i(y)}$. Denoting with $\widetilde{B}_y^{i(y)}$ the ball $B(y, r_y)$, $y \in \partial V$, such that (D) is satisfied, it follows that the collection $\{B_y^{i(y)}\}_{y \in \Omega_1}$ and $\{\widetilde{B}_y^{i(y)}\}_{y \in \Omega_2}$ cover $\Omega_1\backslash\bigcup_{y \in \Omega_2} \widetilde{B}_y^{i(y)}$ and Ω_2, respectively. As a consequence

$$\overline{\Omega} = \Omega_1 \cup \Omega_2 \subset \Big(\bigcup_{y \in \Omega_1} B_y^{i(y)} \Big) \cup \Big(\bigcup_{y \in \Omega_2} \widetilde{B}_y^{i(y)} \Big),$$

and hence, via a compactness argument, we find $y_1, \ldots, y_{n_1} \in \Omega_1$, $y_{n_1+1}, \ldots, y_{n_1+n_2} \in \Omega_2$, $n_j \in \mathbf{N}$, $j = 1,2$, such that:

$$\overline{\Omega} \subset \Big[\Big(\bigcup_{j=1}^{n_1} B_{y_j}^{i(y_j)} \Big) \cup \Big(\bigcup_{k=1}^{n_2} \widetilde{B}_{y_{n_1+k}}^{i(y_{n_1+k})} \Big) \Big]. \tag{51}$$

Now, for $\nu = 1, 2, 3$, $j = 1, \ldots, n_1$, $k = 1, \ldots, n_2$, we set $B^\nu = \bigcup_{i(y_j) = \nu} B_{y_j}^{i(y_j)}$, $\widetilde{B}^\nu = \bigcup_{i(y_{n_1+k}) = \nu} \widetilde{B}_{y_{n_1+k}}^{i(y_{n_1+k})}$, so that (51) can be rewritten as

$$\overline{\Omega} \subset \left[\left(\bigcup_{\nu=1}^{3} B^\nu \right) \cup \left(\bigcup_{\nu=1}^{3} \widetilde{B}^\nu \right) \right]. \tag{52}$$

From the definition of B_y^k, $y \in \Omega_1$, and $\widetilde{B}_y^k$, $y \in \Omega_2$, $k \in \{1,2,3\}$, for any fixed $\nu \in \{1,2,3\}$ we have $D_{x_\nu} \rho(z) \neq 0$ for every $z \in \overline{B}^\nu \cup (\widetilde{B}^\nu \setminus \bigcap_{i(y_{n_1+k}) = \nu,\, k=1,\ldots,n_2} U_{y_{n_1+k}}^{i(y_{n_1+k})})$, where $U_{y_{n_1+k}}^{i(y_{n_1+k})}$ is the subset of three-dimensional Lebesgue measure equal to zero appearing on (D) and related to y_{n_1+k}, $k = 1, \ldots, l_2$. In particular, the last assertion follows from the inclusion

$$\bigcup_{j=1}^{n_0} (A_j \setminus B_j) \subset \left(\bigcup_{j=1}^{n_0} A_j \right) \setminus \left(\bigcap_{j=1}^{n_0} B_j \right),$$

which is satisfied for any collection of sets $\{A_j\}_{j=1}^{n_0}$ and $\{B_j\}_{j=1}^{n_0}$, $n_0 \in \mathbf{N}$.

For the sake of brevity, from now on, for every $\nu = 1, 2, 3$, we denote by U^ν the subset (of three-dimensional Lebesgue measure equal to zero) $\bigcap_{i(y_{n_1+k}) = \nu,\, k=1,\ldots,n_2} U_{y_{n_1+k}}^{i(y_{n_1+k})}$. Therefore, for every $g \in L^s(l_3, l_4)$, from (52) we get

$$\|g \circ \rho\|_{L^s(\Omega)}^s \leq \sum_{\nu=1}^{3} \left(\|g \circ \rho\|_{L^s(\Omega \cap B^\nu)}^s + \|g \circ \rho\|_{L^s(\Omega \cap \widetilde{B}^\nu)}^s \right)$$

$$= \sum_{\nu=1}^{3} \left(\|g \circ \rho\|_{L^s(\Omega \cap B^\nu)}^s + \|g \circ \rho\|_{L^s(\Omega \cap (\widetilde{B}^\nu \setminus U^\nu))}^s \right). \tag{53}$$

Let now $\nu \in \{1,2,3\}$ be fixed and, for $j, k \in \{1,2,3\} \setminus \{\nu\}$, $j \neq k$, consider the change of variables $F^\nu : (x_\nu, x_j, x_k) \to (\xi_\nu, \xi_j, \xi_k)$, where $(x_\nu, x_j, x_k) \in \Omega \cap (B^\nu \cup (\widetilde{B}^\nu \setminus U^\nu))$ and $(\xi_\nu, \xi_j, \xi_k) = (\rho(x_\nu, x_j, x_k), x_j, x_k)$. Since $D_{x_\nu} \rho(z) \neq 0$ for every $z \in B^\nu \cup (\widetilde{B}^\nu \setminus U^\nu)$, from the inverse function theorem we get $(x_\nu, x_j, x_k) = (\lambda_\nu(\xi_\nu, \xi_j, \xi_k), \xi_j, \xi_k) =: G^\nu(\xi_1, \xi_2, \xi_3)$ for every $(\xi_\nu, \xi_j, \xi_k) \in F^\nu(\Omega \cap (B^\nu \cup (\widetilde{B}^\nu \setminus U^\nu)))$. Of course, the set $F^\nu(\Omega \cap (B^\nu \cup (\widetilde{B}^\nu \setminus U^\nu)))$ is not easily characterized, but we can surely say that it is contained in $[l_3, l_4] \times \Pi_{j,k}(\overline{\Omega})$, where $\Pi_{j,k}(\overline{\Omega})$, $j, k = 1, 2, 3$, $j \neq k$, denotes the continuous projection of $\overline{\Omega}$ on the plane $x_j x_k$ acting in the following way:

$$\Pi_{j,k}(x_\nu, x_j, x_k) = (0, x_j, x_k).$$

In particular, $\overline{\Omega}$ being compact, $\Pi_{j,k}(\overline{\Omega})$, $j, k = 1, 2, 3$, is a compact (and hence measurable) subset of $\mathbf{R}^2$ having two-dimensional Lebesgue measure $m_2(\Pi_{j,k}(\overline{\Omega}))$ bounded from above by $m_2(\Pi_{j,k}(Q))$, Q being any three-dimensional cube containing $\overline{\Omega}$. Therefore, denoting by $A(l)$ the set $\Pi_{j,k}(\{x \in \overline{\Omega} \cap V : \rho(x) = l\})$, $l \in [l_3, l_4]$, $A(l)$ turns out to be a compact subset of $\Pi_{j,k}(\overline{\Omega})$, since the level set $\{x \in \overline{\Omega} \cap V : \rho(x) = l\}$, $l \in [l_3, l_4]$, is compact and $\Pi_{j,k}$ is a continuous map. As a consequence $A(l)$, $l \in [l_3, l_4]$, is a two-dimensional Lebesgue measurable subset of $\Pi_{j,k}(\overline{\Omega})$ and hence, using (E) and denoting by $\mathcal{J}F^\nu$ the Jacobian of F^ν, from Fubini's theorem we get

$$\|g \circ \rho\|_{L^s(\Omega \cap B^\nu)}^s + \|g \circ \rho\|_{L^s(\Omega \cap (\widetilde{B}^\nu \setminus U^\nu))}^s$$

$$= \int_{l_3}^{l_4} |g(\xi_\nu)|^s \, d\xi_\nu \int_{A(\xi_\nu)} |\det \mathcal{J}F^\nu(G^\nu(\xi_\nu, \xi_j, \xi_k))| \, d\xi_j \, d\xi_k \leq \widetilde{C} m_2(A(\xi_\nu)) \|g\|_{L^s(l_3, l_4)}^s$$

$$\leq \widetilde{C} m_2(\Pi_{j,k}(\overline{\Omega})) \|g\|_{L^s(l_3, l_4)}^s, \qquad \forall g \in L^s(l_3, l_4). \tag{54}$$

Now, since ν was any index belonging to $\{1,2,3\}$, by replacing (54) in (53) and summing up on ν we obtain (50). The proof is complete. $\square$

With the Lemma 5.4 at hand, we can now prove Theorem 5.3.

Proof of Theorem 5.3. First, for any $p \in (3, +\infty)$ let us choose the Banach spaces X, $X_{1/2}$, $\mathcal{D}(A)$, $\mathcal{D}(B)$, $\mathcal{D}(C)$, Y and Y_1 according to the rule

$$\begin{cases} X = L^p(\Omega), \quad X_{1/2} = W_{D,N}^{1,p}(\Omega), \\ \mathcal{D}(A) = W_{D,N}^{2,p}(\Omega), \quad \mathcal{D}(B) = W^{2,p}(\Omega), \quad \mathcal{D}(C) = W^{1,p}(\Omega), \\ Y = L^p(l_3, l_4), \quad Y_1 = W^{1,p}(l_3, l_4), \end{cases} \tag{55}$$

where the spaces $W_{\mathrm{D,N}}^{1,p}(\Omega)$ are defined by (47) with $\gamma = 1/2$. Then, $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ being defined by (3) and λ_0 being a large enough (fixed) positive constant, we define the operators A, B, C with domains $\mathcal{D}(A)$, $\mathcal{D}(B)$, $\mathcal{D}(C)$ as follows

$$Au = (\mathcal{A} - \lambda_0 I)u, \ u \in \mathcal{D}(A)\,; \quad Bu = \mathcal{B}u, \ u \in \mathcal{D}(B)\,; \quad Cu = \mathcal{C}u, \ u \in \mathcal{D}(C)\,. \tag{56}$$

(55) and (56) ensure that (H2) is satisfied, since (cf. [8]) $X_{1/2} = (X, \mathcal{D}(A))_{1/2,p} = \mathcal{D}_A(1/2, p)$. For the brevity's sake, we refer to [3, pagg. 87–90] for the proof that (H1), (H3), (H4) and (H6)–(H10) are satisfied, and, instead, we turn our attention to (H5) where Lemma 5.4 plays a fundamental role. Indeed, to show that $\mathcal{M}$ is a continuous bilinear operator from $L^p(l_3, l_4) \times W^{1,p}(\Omega)$ to $L^p(\Omega)$ and from $W^{1,p}(l_3, l_4) \times L^p(\Omega)$ to $L^p(\Omega)$ we proceed as follows. First, using Lemma 5.4 and the Sobolev embedding $W^{1,p}(\Omega) \hookrightarrow C^{1-3/p}(\overline{\Omega})$, $p \in (3, +\infty)$, for every pair $(q, w) \in L^p(l_3, l_4) \times W^{1,p}(\Omega)$ we find:

$$\|\mathcal{M}(q,w)\|_{L^p(\Omega)} \le \|q \circ \rho\|_{L^p(\Omega)}\|w\|_{C(\overline{\Omega})} \le C_3(\rho, \Omega, p)\|q\|_{L^p(l_3,l_4)}\|w\|_{W^{1,p}(\Omega)}\,,$$

where the positive constant $C_3(\rho, \Omega, p)$ depends on ρ, Ω and p, only, and is given by the product of the constant $C_1(\rho, \Omega)$ in (50) with the constant $C_2(p, \Omega)$ such that $\|w\|_{C(\overline{\Omega})} \le C_2(p, \Omega)\|w\|_{W^{1,p}(\Omega)}$. Similarly, using $W^{1,p}(l_3, l_4) \hookrightarrow C^{1-1/p}([l_3, l_4])$, $p \ge 1$, for any pair $(q, w) \in W^{1,p}(l_3, l_4) \times L^p(\Omega)$ we get:

$$\|\mathcal{M}(q,w)\|_{L^p(\Omega)} \le C_4(p, \Omega)\|q\|_{W^{1,p}(l_3,l_4)}\|w\|_{L^p(\Omega)}\,.$$

Hence, the bilinearity of $\mathcal{M}$ being obvious, we have that assumption (H5) is satisfied.

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